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The influence of regenerative losses on the performance of a Fermi Ericsson refrigeration cycle

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Abstract

The performance of the Ericsson refrigeration cycle using an ideal Fermi gas as the working substance, which is simply referred to as the Fermi Ericsson refrigeration cycle, is examined, based on the equation of state of an ideal Fermi gas. The inherent regenerative losses and the coefficient of performance (COP) of the cycle are calculated. Three special cases are discussed in detail. It is found that under the conditions of low and high temperatures, the COP of the cycle is only a function of temperature, while under other conditions, the COP of the cycle is not only a function of temperature, but also is dependent on the pressures and other parameters. The results obtained here may reveal the general performance characteristics of the Fermi Ericsson refrigeration cycle.

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1. Introduction

The Ericsson refrigeration cycle is one of the important refrigeration cycle modes with regeneration. According to the theory of classical thermodynamics, the Ericsson refrigeration cycle using an ideal classical gas as the working substance may possess the condition of perfect regeneration through the use of a reversible regenerator. Its coefficient of performance (COP) is equal to that of a reversible Carnot refrigeration cycle. However, when the gas temperature is low enough or density is high enough, the ideal gas will deviate from classical gas behaviour and quantum degeneracy of the gas will become important [1, 2]. Thus, when the working substance is a quantum gas, the Ericsson refrigeration cycle will have some new performance characteristics to be researched.

Unlike a reversible Carnot refrigeration cycle, the performance of the Ericsson refrigeration cycle is dependent on the properties of the working substance [3–5]. The working substances of a quantum Ericsson cycle may be ideal quantum gases, harmonic oscillator systems, spin systems, and so on. For different working substances, the performance of the cycle will be different. To our knowledge, the performance of the Ericsson refrigeration cycle using an ideal Fermi gas as the working substance has not been investigated, although the Ericsson power cycles working with quantum gases have been examined [2].

In this paper, the thermodynamic properties of an ideal Fermi gas in some important processes are analysed, based on the equation of state of an ideal Fermi gas. Using the properties, the inherent regenerative losses of the Fermi Ericsson refrigeration cycle are calculated. The influence of the regenerative losses on the COP is analysed. Some special cases are discussed in detail.

2. Thermodynamic properties of an ideal Fermi gas

According to quantum statistics, the expressions of the pressure, number density, internal energy and entropy for an ideal Fermi gas are given by [6]

$$P = \frac{gkT}{\lambda^3} f_{5/2}(z) = nkTF(z) \tag{1}$$

$$n = \frac{N}{V} = \frac{g}{\lambda^3} f_{3/2}(z) \tag{2}$$

$$U = \tfrac{3}{2} NkTF(z) \tag{3}$$

and

$$S = Nk\left[\tfrac{5}{2}F(z) - \ln z\right] \tag{4}$$

respectively, where g is a weight factor that arises from the 'internal structure' of the particles (such as spin),

$$\lambda = h/(2\pi mkT)^{1/2} \tag{5}$$

where λ, m and N are, respectively, the mean thermal wavelength, rest mass and total number of the particles, V, $z = \exp(\mu/kT)$ and μ are, respectively, the volume, fugacity and chemical potential of the gas, h is the Planck constant, k is the Boltzmann constant, T is the gas temperature,

$$f_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}\,\mathrm{d}x}{z^{-1}\,\mathrm{e}^x + 1} \tag{6}$$

is called the Fermi function, $\Gamma(n)$ is the gamma function, and

$$F(z) = \frac{f_{5/2}(z)}{f_{3/2}(z)} \tag{7}$$

is called the corrected factor. When $F(z) = 1$, an ideal Fermi gas becomes an ideal classical gas.

Using equations (1)–(3), the heat capacity at constant pressure is given by

$$C_P = \frac{5}{2} Nk \frac{\mathrm{d}}{\mathrm{d}T}[TF(T, P)] \tag{8}$$

where $F(T, P)$ is a function of temperature and pressure. From equations (4) and (8), one can find that the amounts of heat exchange in an isothermal and an isobaric process are, respectively, determined by

$$Q_{if} = T(S_f - S_i) = NkT\left\{\tfrac{5}{2}[F(z_f) - F(z_i)] - (\ln z_f - \ln z_i)\right\} \tag{9}$$

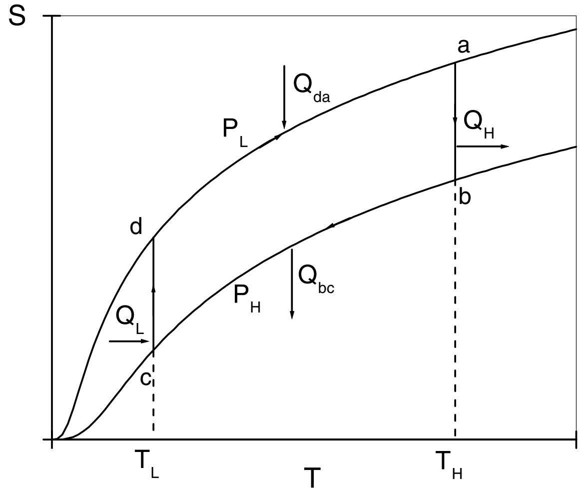

Figure 1. The entropy–temperature diagram of a Fermi Ericsson refrigeration cycle.

and

$$Q_{if}^{P} = \int_{i}^{f} C_P(T, P)\,\mathrm{d}T = \frac{5}{2}Nk[T_f F(T_f, P) - T_i F(T_i, P)] \tag{10}$$

where the subscripts i and f refer to the initial and final states.

With the help of equations (9) and (10), we can analyse the performance characteristics of a quantum Ericsson refrigeration cycle using an ideal Fermi gas as the working substance.

3. A Fermi Ericsson refrigeration cycle

When an ideal gas is used as the working substance, an Ericsson refrigeration cycle is composed of two isothermal and two isobaric processes. Its entropy–temperature diagram is shown in figure 1, where Q_L and Q_H are the amounts of heat exchange between the working substance and the heat reservoirs at temperatures T_L and T_H during the two isothermal processes, Q_{bc} and Q_{da} are the amounts of heat exchange between the working substance and the regenerator during two isobaric processes, and P_L and P_H are the low and high pressures. All heats Q_L, Q_H, Q_{bc} and Q_{da} are positive. In order to improve the performance of the cycle, a regenerator is often applied in two isobaric processes. When Q_{bc} is not equal to Q_{da}, the regenerative losses will be inevitable in spite of the use of a regenerator. For the sake of convenience, the Ericsson refrigeration cycle mentioned above is simply referred to as the Fermi Ericsson refrigeration cycle.

Using equations (9) and (10), the amounts of heat exchange in the various processes of a Fermi Ericsson refrigeration cycle may be expressed as

$$Q_L = \tfrac{5}{2}NkT_L[F(z_d) - F(z_c)] - NkT_L(\ln z_d - \ln z_c) \tag{11}$$

$$Q_H = \tfrac{5}{2}NkT_H[F(z_a) - F(z_b)] - NkT_H(\ln z_a - \ln z_b) \tag{12}$$

$$Q_{da} = \int_{T_L}^{T_H} C_P(T, P_L)\,\mathrm{d}T = \frac{5}{2}Nk[T_H F(z_a) - T_L F(z_d)] \tag{13}$$

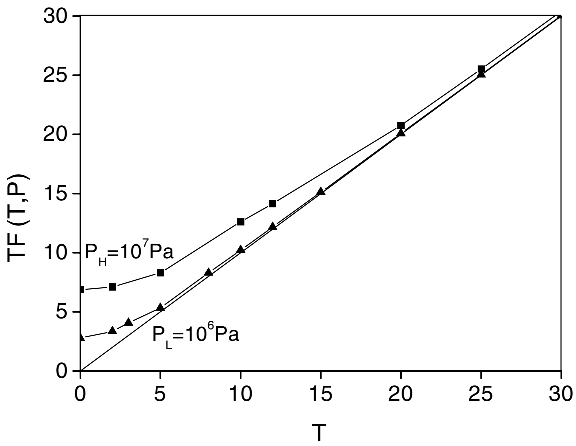

Figure 2. The function $TF(T, P)$ versus temperature curves for two different pressures. The square curve is obtained for a constant high pressure P_H while the triangle curve is obtained for a constant low pressure P_L.

and

$$Q_{bc} = \int_{T_L}^{T_H} C_P(T, P_H)\,\mathrm{d}T = \frac{5}{2}Nk[T_H F(z_b) - T_L F(z_c)] \tag{14}$$

respectively, where z_a, z_b, z_c and z_d are the fugacities of the gas in the states a, b, c and d shown in figure 1. Using equations (11)–(14), we can obtain the regenerative losses and the input work per cycle as

$$\Delta Q = Q_{bc} - Q_{da} = \tfrac{5}{2}Nk\{T_H[F(z_b) - F(z_a)] + T_L[F(z_d) - F(z_c)]\} \tag{15}$$

and

$$W = Q_H - Q_L + Q_{bc} - Q_{da} = NkT_H(\ln z_b - \ln z_a) + NkT_L(\ln z_d - \ln z_c). \tag{16}$$

In order to expound the characteristics of two regenerative processes in the cycle, we plot the function $TF(T, P)$ versus temperature T curves for two different pressures, as shown in figure 2. Using equation (9) and comparing the slopes of the curves in figure 2, we can obtain the following relation:

$$C_P(T, P_H) < C_P(T, P_L). \tag{17}$$

Thus, it is seen from equations (13)–(15) and (17) that $\Delta Q = Q_{bc} - Q_{da} < 0$. This implies the fact that the amount of heat exchange Q_{bc} transferred into the regenerator in one regenerative process is smaller than that of heat exchange Q_{da} transferred from the regenerator in the other regenerative process. The inadequate heat in the regenerator per cycle can only be compensated from the hot reservoir in a timely manner, so that the state of the working substance returns to the original state after each cycle. If not, the temperature of the regenerator would be changed such that the regenerator would not be operated normally. Although there are regenerative losses, the refrigeration heat Q_L per cycle is unvarying. From equations (11) and (16), we obtain the COP of a Fermi Ericsson refrigeration cycle as

$$\varepsilon = \frac{Q_L}{W} = \frac{\frac{5}{2}T_L[F(z_d) - F(z_c)] - T_L(\ln z_d - \ln z_c)}{T_H(\ln z_b - \ln z_a) + T_L(\ln z_d - \ln z_c)}. \tag{18}$$

Using equations (17) and (18), we can analyse the regenerative characteristics and the COP of a quantum Ericsson refrigeration cycle using an ideal Fermi gas as the working substance.

4. Three special cases

Case 1. Under the low-temperature and high-density condition (i.e. the condition of strong gas degeneracy), the natural logarithm of the fugacity and the corrected factor may be expanded to the first approximation (a detailed derivation is given in appendix A), i.e.

$$\ln z = \frac{T_F(P)}{T} - \frac{\pi^2}{4}\frac{T}{T_F(P)} \tag{19}$$

and

$$F(T, P) = \frac{2T_F(P)}{5T} + \frac{\pi^2 T}{10T_F(P)} \tag{20}$$

where $T_F(P) = AP^{2/5}$ is the Fermi temperature and $A = (15\pi^2\hbar^3)^{2/5}/(2km^{3/5})$. By substituting equations (19) and (20) into equations (15) and (18), the regenerative losses and the COP may be, respectively, simplified as

$$\Delta Q = Q_{bc} - Q_{da} = \frac{\pi^2}{4A}Nk\left(T_H^2 - T_L^2\right)\left(P_H^{-2/5} - P_L^{-2/5}\right) < 0 \tag{21}$$

and

$$\varepsilon = \frac{Q_L}{W} = \frac{T_L}{T_H - T_L}\frac{2T_L}{T_H + T_L} < \varepsilon_c \tag{22}$$

where $\varepsilon_c = T_L/(T_H - T_L)$ is the COP of a reversible Carnot refrigeration cycle. In such a case, the COP is only a function of temperature and independent of any other parameters. It is always less than ε_c and decreases quickly as the ratio of the temperatures of two heat reservoirs T_H/T_L increases.

Case 2. Under the high-temperature and low-density condition (i.e. the condition of weak gas degeneracy), the fugacity and the corrected factor may be derived to the first approximation (a detailed derivation is given in appendix B), i.e.

$$z = 4\sqrt{2}BP/T^{5/2}(1 + BP/T^{5/2}) \tag{23}$$

and

$$F(T, P) = 1 + BP/T^{5/2} \tag{24}$$

where $B = (2\pi\hbar^2/m)^{3/2}/(4\sqrt{2}gk^{5/2})$. By substituting equations (23) and (24) into equations (15) and (18), the regenerative losses and the COP may be, respectively, simplified as

$$\Delta Q = \tfrac{5}{2}NkB(P_H - P_L)\left(1/T_H^{3/2} - 1/T_L^{3/2}\right) < 0 \tag{25}$$

and

$$\varepsilon = \frac{T_L \ln(P_H/P_L) - \frac{3}{2}\left(B/T_L^{3/2}\right)(P_H - P_L)}{(T_H - T_L)\ln(P_H/P_L) + B(P_H - P_L)\left[\left(1/T_H^{3/2}\right) - \left(1/T_L^{3/2}\right)\right]} < \varepsilon_c. \tag{26}$$

When ^{3}He gas is chosen as the ideal Fermi gas, equation (26) may be used to generate the curves of the ratio of the COPs, $\varepsilon/\varepsilon_c$, varying with the temperature ratio, $\tau = T_H/T_L$, for three different pressure ratios, $r_P = P_H/P_L$, as shown in figure 3. In this figure, $T_H = 177$ K and $P_H = 10^7$ Pa are given, τ only depends on T_L, and r_P varies with P_L. Under the condition of $T_H > T_L \gg T_F(P_H) = 17.18$ K, the working gas is always in the classical gas state throughout the cycle. Therefore, the COP of an Ericsson refrigeration cycle is approximately equal to that of a Carnot refrigeration cycle.

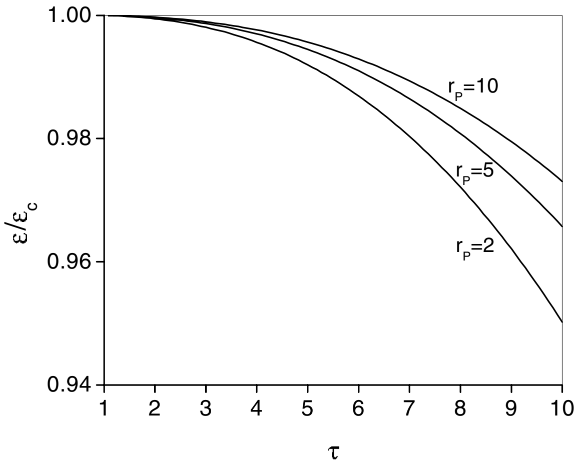

Figure 3. The curves of the ratio of the COPs varying with the temperature ratio for three different pressure ratios. The pressure $P_H = 10^7$ Pa and temperature $T_H = 177$ K are adopted.

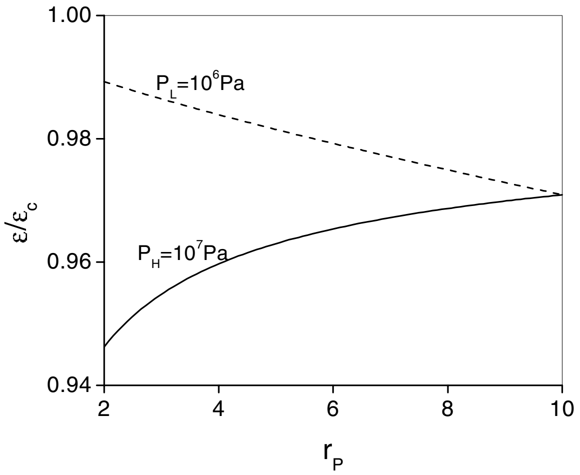

Figure 4. The curves of the ratio of the COPs varying with the pressure ratio. The solid curve is obtained for a constant high pressure P_H while the dashed curve is obtained for constant low pressure P_L. The temperatures $T_H = 177$ K and $T_L = 17.18$ K are adopted.

When the temperature is high enough, $\frac{3}{2}\left(B/T_L^{3/2}\right)(P_H - P_L)/\ln(P_H/P_L) \ll 1$ and $B(P_H - P_L)\left[\left(1/T_H^{3/2}\right) - \left(1/T_L^{3/2}\right)\right]/\ln(P_H/P_L) \ll 1$. Equation (26) is simplified as

$$\varepsilon = \frac{T_L}{T_H - T_L} = \varepsilon_c. \tag{27}$$

In this case, an ideal Fermi gas becomes an ideal classical gas. The Ericsson refrigeration cycle may possess the condition of perfect regeneration so that its COP may attain that of a reversible Carnot refrigeration cycle.

Using equation (26), one can also plot the curves of the ratio of the COPs varying with the pressure ratio, as shown in figure 4, where the temperatures of the heat reservoirs are given as $T_L = 17.18$ K and $T_H = 177$ K. Solid and dashed curves correspond to the cases of a constant P_H value ($P_H = 10^7$ Pa) and a constant P_L value ($P_L = 10^6$ Pa), respectively. For a

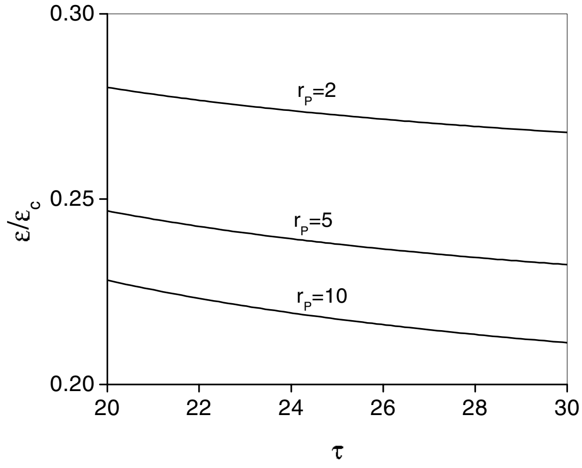

Figure 5. The curves of the ratio of the COPs varying with the temperature ratio for three different pressure ratios. The pressure $P_L = 10^6$ Pa and temperature $T_L = 1$ K are adopted.

constant P_H value, the coefficient of performance increases with the increase of r_P, while for a constant P_L value, the coefficient of performance decreases with the increase of r_P. This implies that the smaller the value of P_L, the larger the COP.

Case 3. Under the condition of $T_H \gg T_F(P_H)$ and $T_L \ll T_F(P_L)$, the fugacity and corrected factor may be, respectively, expressed as

$$z(T_H, P) = 4\sqrt{2}BP\big/T_H^{5/2}\left(1 + BP\big/T_H^{5/2}\right) \tag{28}$$

$$F(T_H, P) = 1 + BP\big/T_H^{5/2} \tag{29}$$

$$\ln z(T_L, P) = \frac{AP^{2/5}}{T_L} - \frac{\pi^2}{4}\frac{T_L}{AP^{2/5}} \tag{30}$$

and

$$F(T_L, P) = \frac{2AP^{2/5}}{5T_L} + \frac{\pi^2 T_L}{10AP^{2/5}}. \tag{31}$$

Substituting equations (28) and (31) into equation (15) and (18), the regenerative losses and the COP may be, respectively, given by

$$\Delta Q = NkA\left(P_L^{2/5} - P_H^{2/5}\right) < 0 \tag{32}$$

and

$$\varepsilon = \frac{(\pi^2/2A)T_L^2\left(P_L^{-2/5} - P_H^{-2/5}\right)}{T_H \ln(P_H/P_L) - A\left(P_H^{2/5} - P_L^{2/5}\right)} \tag{33}$$

which may be used to generate some performance characteristic curves, as shown in figures 5 and 6.

Figure 5 shows the curves of the ratio of the COPs varying with the temperature ratio for three different pressure ratios. In the figure, the value of T_L is given as 1 K and τ depends on T_H only. Similarly, the value of P_L is given as 10^6 Pa and the different values of r_P

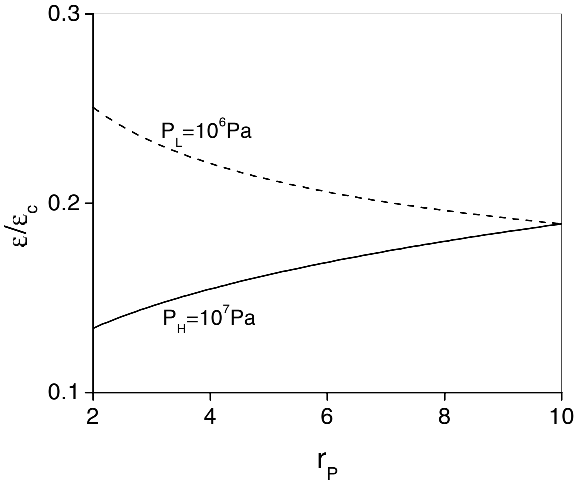

Figure 6. The curves of the ratio of the COPs varying with the pressure ratio. The solid curve is obtained for a constant high pressure P_H while the dashed curve is obtained for a constant low pressure P_L. The temperatures $T_H = 177$ K and $T_L = 1$ K are adopted.

correspond to different P_H values. Under the conditions of $T_H \gg T_F(P_H) = 17.18$ K and $T_L \ll T_F(P_L) = 6.97$ K, the working gas is in the classical gas state for high isothermal processes while it is in the degeneracy state for low isothermal processes. For given values of P_L and T_L, the COP decreases with the increase of τ for different pressure ratios.

Figure 6 shows the curves of the ratio of the COPs varying with the pressure ratio. The temperatures of the heat reservoirs in the cycle are given as $T_L = 1$ K and $T_H = 177$ K. Solid and dashed curves correspond to the cases of a constant P_H value and a constant P_L value, respectively. For a constant P_H value, the coefficient of performance increases with increasing r_P, while for a constant P_L value, the COP decreases with increasing r_P. Obviously, the situations in figure 6 are similar to those in figure 4.

5. Conclusions

We have analysed the performance of the Ericsson refrigeration cycle using an ideal Fermi gas as the working substance. It is found that because there are inherent regenerative losses, the Fermi Ericsson refrigeration cycle may not possess the condition of perfect regeneration. The influence of the inherent regenerative losses on the COP of the cycle is further investigated. The COP under different conditions is derived analytically. In general, the COP is not only a function of temperature, but also is dependent on other parameters. Only if the gas temperature is very low or high enough can the COP be a function of temperature but independent of other parameters. The results obtained may reveal the general performance characteristics of the Fermi Ericsson refrigeration cycle.

Finally, it should be pointed out that we analyse the performance of a Fermi Ericsson refrigeration cycle only in the scope of classical thermodynamics and statistical physics. For a real Ericsson refrigerator using an ideal Fermi gas as the working substance, one has to consider the influence of not only the regenerative losses due to quantum degeneracy of the gas, but also other main irreversible factors often existing in refrigerators. These issues are beyond the scope of this paper and will be further researched.

Acknowledgments

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Appendix A

Under the low-temperature and high-density condition, the Fermi function $f_n(z)$ can be expressed [6] as the asymptotic expansions in powers of the quantity $(\ln z)^{-1}$, i.e.

$$f_{5/2}(z) = \frac{8}{15\sqrt{\pi}}(\ln z)^{5/2}\left[1 + \frac{5\pi^2}{8}(\ln z)^{-2} + \cdots\right] \tag{A1}$$

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}}(\ln z)^{3/2}\left[1 + \frac{\pi^2}{8}(\ln z)^{-2} + \cdots\right]. \tag{A2}$$

Using equations (A1) and (A2), we can obtain the expression of the corrected factor as

$$\begin{aligned} F(z) = \frac{f_{5/2}(z)}{f_{3/2}(z)} &= \frac{2}{5}\ln(z)\frac{1 + \frac{5}{8}\pi^2(\ln z)^{-2} + \cdots}{1 + \frac{1}{8}\pi^2(\ln z)^{-2} + \cdots} \\ &= \frac{2}{5}\ln(z)\left[1 + \frac{\pi^2}{2}(\ln z)^{-2} + \cdots\right]. \end{aligned} \tag{A3}$$

Substituting equation (A1) into equation (1), one has

$$P = \frac{8\pi g}{15}\left(\frac{2m}{h^2}\right)^{3/2}(kT\ln z)^{5/2}\left[1 + \frac{5\pi^2}{8}(\ln z)^{-2} + \cdots\right]. \tag{A4}$$

Using $\mu = kT\ln z$, we can rewrite equation (A4) as

$$\mu = \mu_F\left[1 - \frac{\pi^2}{4}\left(\frac{kT}{\mu}\right)^2 + \cdots\right] \tag{A5}$$

where

$$\mu_F = \left(\frac{15P}{8\pi g}\right)^{2/5}\left(\frac{h^2}{2m}\right)^{3/5} \tag{A6}$$

is called the Fermi energy. From equation (A5), we obtain

$$\ln z = \frac{\mu_F}{kT}\left[1 - \frac{\pi^2}{4}\left(\frac{kT}{\mu_F}\right)^2\right]. \tag{A7}$$

Substituting equation (A7) into equation (A3), one has

$$F(T, P) = \frac{2}{5}\frac{\mu_F}{kT} + \frac{\pi^2}{10}\frac{kT}{\mu_F} = \frac{2}{5}\frac{T_F}{T} + \frac{\pi^2}{10}\frac{T}{T_F} \tag{A8}$$

where $T_F(P) = \mu_F/k = AP^{2/5}$ is called the Fermi temperature.

Appendix B

Under the high-temperature and low-density condition, the Fermi function $f_n(z)$ can be expressed [6] as the asymptotic expansion in powers of the quantity (z),

$$f_{5/2}(z) = z - \frac{z^2}{2^{5/2}} + \frac{z^3}{3^{5/2}} - \cdots \tag{B1}$$

$$f_{3/2}(z) = z - \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} - \cdots . \tag{B2}$$

Using equations (B1) and (B2), we can obtain the expression of the corrected factor as

$$F(z) = \frac{f_{5/2}(z)}{f_{3/2}(z)} = 1 + \frac{1}{4\sqrt{2}} z + \cdots . \tag{B3}$$

From equations (1) and (B1), we obtain

$$y = f_{5/2}(z) = z - z^2/2^{5/2} + z^3/3^{5/2} - \cdots \tag{B4}$$

where $y = \lambda^3 P/gkT$.

Now, let

$$z = a_1 y + a_2 y^2 + \cdots . \tag{B5}$$

Substituting equation (B5) into (B4) and comparing the coefficients of the same powers of y, one obtains $a_1 = 1$ and $a_2 = 1/2^{5/2}$. Substituting these coefficients and y into equation (B4), one has

$$z = \frac{\lambda^3 P}{gkT} + \frac{1}{2^{5/2}} \left(\frac{\lambda^3 P}{gkT} \right)^2 + \cdots . \tag{B6}$$

Substituting equation (B6) into equation (B3), we can obtain the first approximation of the corrected factor as

$$F(T, P) = 1 + BP/T^{5/2} \tag{B7}$$

where $B = (2\pi\hbar^2/m)^{3/2}/(4\sqrt{2}gk^{5/2})$. Thus, the first approximation of equation (B6) is given by

$$z(T, P) = 4\sqrt{2}BP/T^{5/2}(1 + BP/T^{5/2}). \tag{B8}$$

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